# HAUSDORFF MEANS
# AND THE GIBBS PHENOMENON([1])

BY

JONAH MANN

1. **Introduction.** The Hausdorff means of the sequence $S_0, S_1, S_2, \cdots$ are defined by

$$(1) \qquad \sigma_n = \sum_{k=0}^{n} \binom{n}{k} S_k \int_0^1 t^k (1-t)^{n-k} dg(t),$$

$n = 0, 1, 2, \cdots$, where $g(t)$ is of bounded variation in $0 \le t \le 1$. If, in addition, $g(t)$ satisfies $g(0+) = g(0) = 0$ and $g(1) = 1$, the transform (1) is regular [10, p. 119], i.e., $\sigma_n \to S$ whenever $S_n \to S$ as $n \to \infty$.

The Gibbs phenomenon may be described as follows. Suppose that a sequence of functions $\{f_n(x)\}$ converges to a function $f(x)$ for $x_0 - h \le x \le x_0 + h$ and that $f(x_0 +)$ and $f(x_0 -)$ exist. If, when $n \to \infty$ and $x \to x_0$ independently,

$$\limsup f_n(x) > \max \left[ f(x_0 +), f(x_0 -) \right]$$

or

$$\liminf f_n(x) < \min \left[ f(x_0 +), f(x_0 -) \right],$$

then $\{f_n(x)\}$ is said to exhibit the Gibbs phenomenon at the point $x = x_0$.

O. Szász [9] proved that the Hausdorff means, corresponding to a particular $g(t)$, of the sequence of partial sums of the Fourier series

$$(2) \qquad \sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{1}{2} (\pi - x), \qquad 0 < x < 2\pi,$$

will exhibit the Gibbs phenomenon at the point $x = 0$ if and only if

$$(3) \qquad \int_0^1 \int_0^A \frac{\sin ty}{y} dy \, dg(t) > \frac{\pi}{2}$$

for some $A > 0$.

It is known [12, p. 61] that the occurrence of the Gibbs phenomenon for the Hausdorff means of a Fourier series at a point of ordinary discontinuity is equivalent to its presence for the special series (2) at $x = 0$.

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The purpose of this paper is to find classes of functions, $g(t)$, which satisfy the inequality in (3). At first, we generalize some known results of Livingston [4] and Newman [7]. The class of functions $g(t) = t^n, n > 0$, whch bears a close resemblance to the $g(t)$ for the Cesàro and Hölder means [9], provides many counterexamples. It, like the Cesàro and Hölder means, has a constant which breaks up the interval $0 < n < \infty$ into two intervals such that $g(t) = t^n$ presents the phenomenon for values of $n$ in one interval whereas for values of $n$ in the other, it does not.

The main result of this paper states that $g(t)$, with $g(t)/t$ belonging to $L^2(0, 1)$, will exhibit the Gibbs phenomenon if and only if $H(x) = \int_{-\infty}^{\infty} f(t + x)g(t)/t \, dt$ is not positive definite. The conditions on $f(t)$ are liberal enough to allow an ample reservoir of functions from which to choose. Although it is difficult to determine whether or not a function is positive definite, and especially the function we have provided, it is always possible to find an $f(t)$, when $g(t)$ exhibits the phenomenon, for which $H(x)$ is negative for $x = 0$ and, therefore, not positive definite. A number of theorems illustrating this point are included.

**2. Some generalizations.** In what follows, $g(t)$ will be used exclusively to denote a function of bounded variation in $0 \leq t \leq 1$ with $g(0+) = g(0) = 0$ and $g(1) = 1$, and $h(t)$ will equal $g(t)/t$. We find it useful to let $g(t)$ be normalized in $[0, 1]$. Furthermore, $g(t)$ is defined outside this interval by $g(t) = 1$ for $t > 1$ and $g(-t) = g(t)$.

As was mentioned previously, the existence of the Gibbs phenomenon for the Hausdorff summability method given by $g(t)$ is equivalent to the statement in (3). Integration by parts gives

$$\int_0^1 \int_0^A (\sin ty)/y \, dy \, dg(t) = \int_0^1 (\sin At)/t \, dt - \int_0^1 \sin At \, \frac{g(t)}{t} \, dt.$$

We have, therefore,

LEMMA 1.   $g(t)$ *will exhibit the Gibbs phenomenon if and only if*

(4) $$F(A) = \int_0^1 \sin At \, \frac{g(t)}{t} \, dt + \int_1^\infty \frac{\sin At}{t} dt < 0$$

*for some $A > 0$.*

D. J. Newman [7] has shown that if the measure $dg(t)$ has at least one mass point and satsifies $\int_0^1 |dg(t)|/t^2 < \infty$, then $g(t)$ exhibits the Gibbs phenomenon. Since $g(t)$ is continuous at $t = 0$, his latter condition implies that $h(t)$ is of bounded variation in $0 \leq t \leq 1$ and that $h(0+) = 0$. Consequently, Newman's result is contained in the following:

THEOREM 1.   *Let $h(t) = g(t)/t$ be of bounded variation in $0 \leq t \leq 1$, and let $h(0+) = h(0) = 0$. $g(t)$ will exhibit the Gibbs phenomenon if it has at least one point of discontinuity.*

**Proof.** We shall show that $F(A) < 0$ for some $A > 0$ by proving

1. $\int_1^y F(A)\, dA = o(\log y)$  as  $y \to \infty$,
2. $\int_1^y |F(A)|\, dA \sim M \log y$,    $M > 0$.

We require the following lemmas; the first two may be proved by integrating by parts.

**LEMMA 2.** $\int_1^\infty (\sin At)/t\, dt = (\cos A)/A + O(1/A^2)$ as $A \to \infty$.

**LEMMA 3.** $F(A) = (1/A) \int_0^1 \cos At\, dh(t) + O(1/A^2)$ as $A \to \infty$, where $h(t)$ is of bounded variation.

**LEMMA 4.** If $h(t)$ is of bounded variation in $0 \le t \le 1$,

$$\int_1^y (1/A) \int_0^1 \cos At\, dh(t)\, dA = o(\log y)$$

as $y \to \infty$.

**Proof.** Let $\{\delta_k\}$ denote the points of discontinuity of $h(t)$, and let $h(t) = j(t) + c(t)$ where

$$j(t) = \sum_{\delta_k \le t} [h(\delta_k +) - h(\delta_k -)].$$

$j(t)$ is of bounded variation and $c(t)$ is continuous and of bounded variation. Lorch and Newman [5] have shown

$$\int_1^y (1/A) \left| \int_0^1 \cos At\, dc(t) \right| dA = o(\log y).$$

Thus, we may consider only

$$\int_1^y (1/A) \int_0^1 \cos At\, dj(t)\, dA = \int_1^y (1/A) \sum_n D_n \cos A\delta_n\, dA,$$

where $D_n = h(\delta_n +) - h(\delta_n -)$. Since $\sum_n |D_n| < \infty$, we may integrate term by term. Given $\varepsilon > 0$, choose $n_0$ so that $\sum_{n \ge n_0} |D_n| < \varepsilon$. Integration by parts gives

$$\left| \int_1^y \frac{\cos A\delta_n}{A} dA \right| \le 2 + \frac{1}{\delta_n}  \text{ when } y \ge 1.$$

Therefore,

$$\left| \int_1^y (1/A) \sum_{n < n_0} D_n \cos A\delta_n\, dA \right| \le \sum_{n < n_0} \left( 2 + \frac{1}{\delta_n} \right) |D_n|.$$

On the other hand,

$$\left| \int_1^y (\cos A\delta_n)/A\, dA \right| \le \log y,$$

so that

$$\left| \int_1^y (1/A) \sum_{n \geq n_0} D_n \cos A\delta_n \, dA \right| \leq \sum_{n \geq n_0} |D_n| \log y \leq \varepsilon \log y.$$

Finally,

$$\int_1^y (1/A) \int_0^1 \cos At \, dj(t) \, dA = o(\log y) + \varepsilon \log y.$$

Since $\varepsilon$ is arbitrary, the proof is complete.

**Proof of 1.** By Lemma 3,

$$\int_1^y F(A) \, dA = \int_1^y (1/A) \int_0^1 \cos At \, dh(t) \, dA + O(1).$$

Therefore, by Lemma 4, $\int_1^y F(A) \, dA = o(\log y)$.

**Proof of 2.** Let $G(A) = \int_0^1 \cos At \, dh(t)$. If we again set $h(t) = c(t) + j(t)$, we split $G(A)$ into $C(A) + J(A)$ where $J(A)$ is almost periodic. It is known [5] that $(1/y) \int_0^y |C(A)| \, dA \to 0$ as $y \to \infty$ whereas $(1/y) \int_0^y |J(A)| \, dA \to M > 0$, where $M$ is the mean value of the positive almost periodic function $|J(A)|$. Therefore, $(1/y) \int_1^y |G(A)| \, dA \to M$.

Integration by parts yields

$$\frac{1}{\log y} \int_1^y \frac{|G(A)|}{A} \, dA \to M.$$

Since

$$\int_1^y |F(A)| \, dA = \int_1^y |G(A)|/A \, dA + O(1),$$

$$\int_1^y |F(A)| \, dA \sim M \log y.$$

The proof is now complete.

The $g(t)$ in the following example satisfies all the conditions in Theorem 1 except for $h(0+) = h(0) = 0$, but does not present the phenomenon. Later, other counterexamples will be given for this theorem and others.

EXAMPLE 1. Let

$$g(t) = \begin{cases} 4t, & 0 \leq t \leq \frac{1}{2}, \\ t, & \frac{1}{2} < t \leq 1. \end{cases}$$

Then, from (4),

$$F(A) \geq \frac{3 - 3 \cos A/2}{A} \geq 0,$$

since

$$\int_A^\infty (\sin t)/t \, dt = (\cos A)/A - \int_A^\infty (\cos t)/t^2 \, dt$$

and

$$\left| \int_A^\infty (\cos t)/t^2 \, dt \right| < 1/A.$$

Consequently, this $g(t)$ does not present the phenomenon.

The following theorem is a generalization of a result of Livingston [4].

THEOREM 2. *Let* $g(t) = tf_i(t)$ *in* $\delta_{i-1} \le t < \delta_i$, $i = 1, 2, \cdots, n$, *where* $\delta_0 = 0$, *and let* $g(t) = tf_{n+1}(t)$ *in* $\delta_n \le t \le \delta_{n+1} = 1$. *Let* $f_1(0) = p$, *and suppose that for each* $i$, $i = 1, 2, \cdots, n+1$, $f_i(t)$ *is absolutely continuous in* $(\delta_{i-1}, \delta_i)$. *Let* $c_i$ *denote the value of the jump of* $g(t)/t$ *at* $t = \delta_i$. *If* $\delta_1, \delta_2, \cdots, \delta_n$, *and* 1 *are linearly independent over the rationals, the Gibbs phenomenon will hold for* $g(t)$ *if* $g'(0) = p < \sum_{i=1}^n |c_i|$. *In particular, if* $g'(0) < 0$, $g(t)$ *need have no discontinuities.*

**Proof.** We may write (4) as

$$F(A) = \sum_{i=1}^{n+1} \int_{\delta_{i_1}}^{\delta_i} \sin At \, f_i(t) \, dt + \int_A^\infty \frac{\sin t}{t} \, dt.$$

Integration by parts plus the Riemann-Lebesgue lemma gives

$$\int_{\delta_{i-1}}^{\delta_i} \sin At \, f_i(t) \, dt = (1/A)[f_i(\delta_{i-1}) \cos A\delta_{i-1} - f_i(\delta_i) \cos A\delta_i] + o(1/A).$$

Thus, by Lemma 2,

$$(5) \qquad F(A) = (1/A)(p + c_1 \cos A\delta_1 + \cdots + c_n \cos A\delta_n) + o(1/A).$$

Assume that the $c_i$ have been rearranged, if necessary, so that for some $m \le n$, $c_1, \cdots, c_m > 0$ and $c_{m+1}, \cdots, c_n < 0$. Livingston [4] and others have shown that if $\delta_1, \cdots, \delta_n$, 1 are linearly independent over the rationals, then given any $\varepsilon > 0$, there exist positive odd integers $x, I_1, \cdots, I_m$ and positive even integers $I_{m+1}, \cdots, I_n$ such that $0 < x\delta_k - I_k < \varepsilon$ for $k = 1, 2, \cdots, n$.

Let $\varepsilon = 1/KN\pi$ where $K$ and $N$ will be defined shortly. Note that for any choice of $K$ and $N$, there are integers $I_k$ such that $K\pi x\delta_k - K\pi I_k < 1/N$. Since $I_k$ is odd for $1 \le k \le m$ and even for $m + 1 \le k \le n$, $c_1 \cos K\pi x\delta_1 + \cdots + c_n \cos K\pi x\delta_n$ can be made arbitrarily close to $- \sum_{i=1}^n |c_i|$ if $K$ is a positive odd integer and $N$ is chosen sufficiently large.

Hence, first choose an $N$ so that, for any choice of (odd) $K$, there exist $x, I_1, \cdots, I_n$ for which

$$- \sum_{i=1}^n |c_i \cos K\pi x\delta_i| + p < 0.$$

Next choose $K$ so large that the $o(1/A)$ term in (5) is smaller than

$$(1/A)\left|p - \sum_{i=1}^{n} |c_i \cos K\pi x\delta_i|\,\right|$$

for all $A \geqq K\pi$. Call $A_0 = K\pi \, x \geqq K\pi$. For this choice of $K$ and $N, F(A_0) < 0$, and the proof is complete.

### 3. Continuous $g(t)$.

THEOREM 3. *Let $h'(t)$ exist and be continuous and of bounded variation in* $0 \leqq t \leqq 1$. $g(t)$ *will exhibit the Gibbs phenomenon if $h(0) = 0$ and $h'(1) \neq -1$.*

**Proof.** This proof is similar to that of Theorem 1, except that here we show $F(A)$ (4) is negative for some $A > 0$ by proving
1. $\int_1^y AF(A)\,dA$ remains bounded as $y \to \infty$,
2. $\int_1^y |AF(A)|\,dA \sim M \log y, M > 0$.
**Proof of 1.** Integration by parts plus Lemma 2 shows that

$$(6) \qquad AF(A) = \int_0^1 \cos At\, h'(t)\,dt + (\sin A)/A + O(1/A^2).$$

Thus,

$$\int_1^y AF(A)\,dA = \int_1^y \int_0^1 \cos At\, h'(t)\,dt\,dA + \int_1^y (\sin A)/A\,dA + \int_1^y O(1/A^2)\,dA$$

$$= I_1 + I_2 + I_3.$$

$I_2$ and $I_3$ remain bounded as $y \to \infty$. Since $h'(t)$ is of bounded variation, we have, by Fubini's theorem, that

$$I_1 = \int_0^1 \int_1^y \cos At\, h'(t)\,dA\,dt$$

$$= \int_0^1 \frac{\sin yt}{t} h'(t)\,dt - \int_0^1 \frac{\sin t}{t} h'(t)\,dt,$$

each of which remains bounded as $y \to \infty$.

**Proof of 2.**

$$\int_0^1 \cos At\, h'(t)\,dt = A^{-1}\int_0^1 h'(t)\,d\sin At$$

$$= h'(1)(\sin A)/A - A^{-1}\int_0^1 \sin At\, dh'(t).$$

Thus, (6) becomes

$$AF(A) = [h'(1) + 1](\sin A)/A - A^{-1}\int_0^1 \sin At\, dh'(t) + O(1/A^2).$$

Lorch and Newman [5] have shown

$$\int_1^y A^{-1} \left| \int_0^1 \sin At \, dh'(t) \right| dA = o(\log y).$$

(It is here that $h'(t)$ is required to be continuous.) On the other hand,

$$y^{-1} \int_1^y |\sin A| \, dA \to 2/\pi \quad \text{as } y \to \infty$$

so that, as in Theorem 1,

$$\int_1^y |\sin A|/A \, dA \to (2/\pi)\log y.$$

Therefore, if $h'(1) \neq -1$ $(g'(1) \neq 0)$,

$$\int_1^y |AF(A)| dA \sim (2/\pi)[h'(1)+1]\log y.$$

This proves the theorem.

It does seem strange that Theorem 3 requires $g'(1)$ to be unequal to zero $(h'(1) \neq -1)$. We shall see later, however, that this condition is by no means peculiar. We have not found a $g(t)$, with $g'(1) = 0$, which otherwise satisfies the requirements of this theorem and which does not present the phenomenon. Nevertheless, we shall construct such a function for which $F(A)$ is positive for all large $A$. As far as this theorem is concerned, this example will suffice to show the necessity of $g'(1) \neq 0$, for here $F(A)$ is negative for some arbitrarily large values of $A$.

EXAMPLE 2. Let $g(t) = a_0 + a_1 t + \cdots + a_6 t^6$. Now, from (4),

$$F'(A) = \int_0^1 \cos At \, g(t) \, dt - (\sin A)/A.$$

By repeated integrations by parts, we find

$$F'(A) = (1/A^2)[g'(1)\cos A - g'(0)] - (1/A^3)g''(1)\sin A$$

$$+ (1/A^4)[g'''(0) - g'''(1)\cos A] + (1/A^4) \int_0^1 g^{(4)}(t)\cos At \, dt.$$

It is certainly possible to find values of $a_i, i = 0, 1, \cdots, 6$, so that $g(t)$ satisfies the conditions: $g(0) = 0$, $g(1) = 1$, $g'(0) = 0$, $g'(1) = 0$, $g''(1) = 0$, $g'''(0) = -1$, and $g'''(1) = 0$. Therefore, by the Riemann-Lebesgue lemma,

$$F'(A) = -(1/A^4) + O(1/A^5).$$

Thus, for all sufficiently large $A$, $F'(A)$ is negative. Since $F(A) \to 0$ as $A \to \infty$

and $F(A)$ is a decreasing function of $A$ for all large $A$, it follows that $F(A)$ is positive for all large $A$.

It is interesting to note how the properties of the continuity of $g(t)$ and the order of its zero at $t = 0$ may be combined to give sufficient conditions for the existence of the Gibbs phenomenon. The $g(t)$ in Theorem 3 is continuous and is required to have a zero of order greater than or equal to two at $t = 0$. If $g(t)$ is discontinuous, Theorem 1 allows the order of zero to be lowered by one. If, however, the order is precisely one, then discontinuity alone is not enough, as seen in Example 1. In this case, Theorem 2 shows that the linear independence of the points of jump over the rationals is helpful.

The following three theorems, the proofs of which are not given here, are to a great extent dependent upon the behavior of $g(t)$ at $t = 0$.

THEOREM 4. *$g(t)$ will not exhibit the Gibbs phenomenon if it is absolutely continuous and if $g'(t)$ is positive decreasing.*

THEOREM 5. *If $h(t) = g(t)/t$ is of bounded variation in $0 \leqq t \leqq 1$, and if $g'(0) < 0$, then $g(t)$ will present the Gibbs phenomenon.*

THEOREM 6. *Let $g(t) = t^\alpha f(t)$ where $f(0 +) > 0$, $f(t)$ is of bounded variation in $0 \leqq t \leqq 1$, and $0 < \alpha < 1$. Then the corresponding $F(A)$ is positive for all large $A$.*

It is apparent that if a function which satisfies the conditions in Theorem 6 exhibits the Gibbs phenomenon, then the corresponding $F(A)$ will be negative for (some) relatively small values of $A$ only. On the other hand, Theorems 1, 2, and 3 imply that the corresponding $F(A)$ will be negative for some arbitrarily large values of $A$. Although, as far as Fourier series is concerned, there is no distinction between $F(A)$ being negative for small values of $A$ or large values, there is a difference in being able to give classes of functions which exhibit the phenomenon. This may be seen from the following lemma.

LEMMA 5. *Let $F_1(A)$ and $F_2(A)$ denote the corresponding $F(A)$ for the functions $g_1(t)$ and $g_2(t)$, respectively. If $F_1(A_0) < 0$ and if $\left| g_1(t) - g_2(t) \right|$ is sufficiently small, then $F_2(A_0) < 0$.*

**Proof.**

$$F_2(A_0) = F_1(A_0) + \int_0^1 \frac{\sin A_0 t}{t} \left[ g_2(t) - g_1(t) \right] dt.$$

It is possible, therefore, to construct functions, having various properties at $t = 0$ and elsewhere, which will exhibit the phenomenon because they are close to another function which does. Let us use the class of functions $g(t) = t^n$ as an example.

It shall be seen later that $g(t) = t$ does not exhibit the phenomenon while

$g(t) = t^2$ makes $F(A)$ negative for arbitrarily large $A$. Hence, Lemma 5 implies that $g(t) = t^n$ will present the phenomenon when $n$ is sufficiently close to two. We shall show, however, that if $g(t) = t^n$, $1 < n < 2$, makes $F(A)$ negative for some $A > 0$, it does so only for relatively small values of $A$.

This class of functions will be studied in detail in the next section. It is noted here that Theorem 8 shows that the requirements in Theorem 1 that $g(t)$ be discontinuous and in Theorem 3 that $h'(t)$ be continuous and of bounded variation are not necessary conditions. On the other hand, it shows that these theorems are no longer true once these conditions are removed.

**4. A special case.** The Hausdorff means given by $g(t) = t^n$ bear a close resemblance to the $g(t)$ for the Cesàro and Hölder means which are, respectively,

$$g(t) = 1 - (1 - t)^p, \qquad p > 0,$$

and

$$g(t) = \frac{1}{\Gamma(p)} \int_0^t \left(\log \frac{1}{y}\right)^{p-1} dy, \qquad p > 0.$$

It is known [12, p. 110] that a constant $p_0$ exists such that the Cesàro means of order $p$, $(C, p)$, exhibit the Gibbs phenomenon if and only if $p < p_0$, where $0 < p_0 < 1$. Such a constant $p_1$, with $p_0 < p_1 < 1$, exists for the Hölder means also [9]. We proceed to prove a corresponding result for the means given by $g(t) = t^n$.

LEMMA 6.   $g(t) = t$ does not exhibit the Gibbs phenomenon.

**Proof.**   From (4),

$$F'(A) = \int_0^1 t \cos At \, dt - (\sin A)/A = (1/A^2)(\cos A - 1) \leq 0.$$

Therefore, $F(A)$ decreases from $\pi/2$ to $0$ as $A$ increases from $0$ to $\infty$, and is never negative for positive $A$.

LEMMA 7.   $g(t) = t^2$ exhibits the Gibbs phenomenon.

**Proof.**   Theorem 3.

THEOREM 7.   *There is a value of $n = n_0$, $1 < n_0 < 2$, for which $g(t) = t^n$ does not exhibit the Gibbs phenomenon.*

**Proof.**   The proof will be a consequence of the following lemmas.

LEMMA 8.   *Let $F_1(A)$ and $F_2(A)$ denote the corresponding $F(A)$ for the functions $g_1(t)$ and $g_2(t)$, respectively. If $F_1(A) > 0$ when $0 \leq A \leq A_0$ and if $|g_1(t) - g_2(t)|$ is sufficiently small, then $F_2(A) > 0$ for $A \leq A_0$.*

**Proof.**

$$F_2(A) = F_1(A) + \int_0^1 \frac{\sin At}{t} [g_2(t) - g_1(t)] \, dt.$$

Let $g_\alpha(t) = t^{2-\alpha}, 0 \leq \alpha \leq 1$, and denote the corresponding $F(A)$ for each $\alpha$ by $F_\alpha(A)$.

LEMMA 9. *Given $A_0$, let $m_0$ be the least upper bound of all $\alpha$, $0 \leq \alpha \leq 1$, for which $F_\alpha(A) < 0$ for some $A \leq A_0$. Then $F_{m_0}(A) \geq 0$ for all $A \leq A_0$ and $F_{m_0}(A) = 0$ for some $A \leq A_0$.*

**Proof.** If $F_{m_0}(A)$ were negative for some $A \leq A_0$, Lemma 5 would imply that $m_0$ is not the least upper bound. In the same way, Lemma 8 proves the second statement.

LEMMA 10. $m_0 \neq 1$ *for any value of $A_0$.*

**Proof.** The proof of Lemma 6 implies that not only is $F_1(A) \geq 0$ but that $F_1(A) > 0$. Hence, by Lemma 9, $m_0 \neq 1$.

LEMMA 11.

$$\lim_{\alpha \to 1-} (1 - \alpha) \int_0^{3\pi/2} \frac{\cos t}{t^\alpha} dt = 1.$$

**Proof.** Integrating by parts, we find

$$\int_0^{3\pi/2} (\cos t)/t^\alpha dt = (1 - \alpha)^{-1} \int_0^{3\pi/2} t^{1-\alpha} \sin t \, dt.$$

Therefore,

$$\lim_{\alpha \to 1-} (1 - \alpha) \int_0^{3\pi/2} (\cos t)/t^\alpha dt = \int_0^{3\pi/2} \sin t \, dt = 1.$$

LEMMA 12. *There exists an $A_0$ and an $\alpha_0$ such that for all $A \geq A_0$ and all $\alpha \geq \alpha_0$, $F_\alpha(A) > 0$.*

**Proof.**

$$F_\alpha(A) = \int_0^1 t^{1-\alpha} \sin At \, dt + \int_A^\infty (\sin t)/t \, dt.$$

Integrating by parts, we find that

$$\int_0^1 t^{1-\alpha} \sin At \, dt = -(\cos A)/A + A^{-1}(1 - \alpha) \int_0^1 t^{-\alpha} \cos At \, dt$$

and

$$\int_A^\infty (\sin t)/t \, dt = (\cos A)/A + (\sin A)/A^2 - 2 \int_A^\infty (\sin t)/t^3 \, dt.$$

Since

$$2 \left| \int_A^\infty (\sin t)/t^3 dt \right| \leq 1/A^2,$$

$$F_\alpha(A) \geq A^{-1}(1 - \alpha) \int_0^1 t^{-\alpha} \cos At \, dt - 2/A^2.$$

Also, since

$$\int_{3\pi/2}^{A} t^{-\alpha} \cos t \, dt \geq 0,$$

$$F_\alpha(A) \geq (1/A^2)\left[A^\alpha(1 - \alpha) \int_0^{3\pi/2} t^{-\alpha} \cos t \, dt - 2\right].$$

By Lemma 11, an $\alpha_0$ may be chosen so that for all $\alpha$, $\alpha_0 \leq \alpha \leq 1$,

$$(1 - \alpha) \int_0^{3\pi/2} t^{-\alpha} \cos t \, dt > \frac{1}{2}.$$

Now choose an $A_0$ for which $A_0^{\alpha_0} > 4$. Since $A^\alpha$ is an increasing function of both $\alpha$ and $A$, this choice of $A_0$ and $\alpha_0$ gives the desired result.

**Proof of Theorem 7.** If there are values of $\alpha$ arbitrarily close to 1 for which $F_\alpha(A) < 0$ for some $A$, then $A \leq A_0$ by Lemma 12. However, in this case, $m_0 = 1$. This contradicts Lemma 10 and completes the proof of Theorem 7.

Let $\sigma_n^\alpha$ denote the Hausdorff means given by $g(t) = t^\alpha$. We shall show that $\sigma_n^{\alpha-h}$, $h > 0$ and $\alpha - h > 0$, may be expressed as linear means of the sequence $\{\sigma_n^\alpha\}$ with the property that the corresponding matrix $M$ (see [12, p. 74]) is a positive regular matrix. This will then imply (see [12, p. 110]) that if the Gibbs phenomenon does not occur for $g(t) = t^\alpha$ for $\alpha = \alpha_0$, it will not occur for any smaller value of $\alpha$.

From (1),

$$\sigma_n^\alpha = \alpha \sum_{k=0}^{n} \binom{n}{k} S_k \int_0^1 t^{\alpha+k-1}(1 - t)^{n-k} dt$$

$$= \alpha \sum_{k=0}^{n} \binom{n}{k} S_{n-k} \int_0^1 t^k(1 - t)^{\alpha+n-k-1} dt.$$

This last expression is precisely the Cesàro means of order $\alpha (g(t) = 1 - (1 - t)^\alpha)$ of the sequence $S_{n-k}$, $k = 0, 1, \cdots, n$. Therefore,

$$A_n^\alpha \sigma_n^\alpha = \sum_{k=0}^{n} A_k^{\alpha-1} S_k$$

where (see [12, p. 77])

$$A_n^\alpha = \frac{(\alpha + 1) \cdots (\alpha + n)}{n!} = \binom{\alpha + n}{n}.$$

Thus,

$$A_n^\alpha \sigma_n^\alpha = \sum_{k=0}^{n-1} A_k^{\alpha-1} S_k + A_n^{\alpha-1} S_n = A_{n-1}^\alpha \sigma_{n-1}^\alpha + A_n^{\alpha-1} S_n.$$

However,

$$\frac{A_{n-1}^{\alpha}}{A_n^{\alpha}} = \frac{\binom{\alpha+n-1}{n-1}}{\binom{\alpha+n}{n}} = \frac{n}{\alpha+n},$$

and

$$\frac{A_n^{\alpha-1}}{A_n^{\alpha}} = \frac{\binom{\alpha+n-1}{n}}{\binom{\alpha+n}{n}} = \frac{\alpha}{\alpha+n}.$$

Therefore,

$$(7) \qquad \sigma_n^{\alpha} = \frac{n}{\alpha+n}\sigma_{n-1}^{\alpha} + \frac{\alpha}{\alpha+n}\,S_n.$$

Using equation (7), it is easy to show by induction that

$$\sigma_n^{\alpha-h} = \frac{\alpha-h}{\alpha(n+\alpha-h)}\left[(n+\alpha)\sigma_n^{\alpha} + \frac{hn\,\sigma_{n-1}^{\alpha}}{n-1+\alpha-h} + \frac{hn(n-1)\,\sigma_{n-2}^{\alpha}}{(n-1+\alpha-h)(n-2+\alpha-h)}\right.$$
$$\left. + \cdots + \frac{h\,n!\,\sigma_0^{\alpha}}{(n-1+\alpha-h)(n-2+\alpha-h)\cdots(\alpha-h)}\right].$$

Consequently, the elements $a_{nk}$ of the matrix $M$ which determines $\sigma_n^{\alpha-h}$ as linear means of $\sigma_n^{\alpha}$ are given by

$$(8) \qquad a_{nk} = \begin{cases} \dfrac{hA_k^{\alpha-h-1}}{\alpha A_n^{\alpha-h}}, & n \neq k, \\[3mm] \dfrac{(\alpha-h)(\alpha+n)}{\alpha(n+\alpha-h)}, & n = k. \end{cases}$$

Now, by (8),

$$\lim_{n\to\infty} a_{nk} = \lim_{n\to\infty} \frac{hA_k^{\alpha-h-1}}{\alpha A_n^{\alpha-h}} = 0 \quad (\text{see } [12, \text{ p. } 77]),$$

and

$$\sum_{k=0}^{n} a_{nk} = \sum_{k=0}^{n-1} \frac{hA_k^{\alpha-h-1}}{\alpha A_n^{\alpha-h}} + \frac{(\alpha-h)(\alpha+n)}{\alpha(n+\alpha-h)}$$

$$= \frac{hA_{n-1}^{\alpha-h}}{\alpha A_n^{\alpha-h}} + \frac{(\alpha-h)(\alpha+n)}{\alpha(n+\alpha-h)} \quad (\text{see } [12, \text{ p. } 77])$$

$$= \frac{hn}{\alpha(\alpha-h+n)} + \frac{(\alpha-h)(\alpha+n)}{(n+\alpha-h)} = 1.$$

Therefore, the Toeplitz conditions for regularity are satisfied, and the matrix $M$ is a positive regular matrix. Lemma 5 implies that the set of $n$, for which $g(t) = t^n$ exhibits the Gibbs phenomenon, is open. Summing up, we have

THEOREM 8. *There is a constant $n_0$, $1 < n_0 < 2$, such that the Gibbs phenomenon will hold for $g(t) = t^n$ if $n > n_0$ and will not hold if $0 < n \leq n_0$.*

By Lemma 3, when $g(t) = t^{1+\alpha}$ $(h(t) = t^\alpha)$, $0 < \alpha < 1$,

$$F(A) = (\alpha/A) \int_0^1 t^{\alpha-1} \cos At \, dt + O(1/A^2).$$

Applying Bromwich's theorem (in the form used in [5, p. 296]) to the above integral, we get

$$F(A) = \alpha\Gamma(\alpha)(\cos \alpha\pi/2) A^{-(1+\alpha)} + o(A^{-(1+\alpha)}).$$

Therefore, when $n_0 < n < 2$, the corresponding $F(A)$ is positive for all large $A$, even though $g(t) = t^n$ presents the phenomenon.

It is well known (e.g., see [3, p. 264]) that the Cesàro and Hölder means of like order are equivalent. In spite of this, the respective constants, which determine for what orders the Gibbs phenomenon will hold, are unequal. The mere fact, then, that two summability methods are equivalent does not necessarily imply that the phenomenon will hold simultaneously for both. The following theorem lends emphasis to this assertion.

THEOREM 9. *The Hausdorff means given by $g(t) = t^\alpha$, $\alpha > 0$, are equivalent to each other.*

**Proof.** Let $\mu_{n,i}$ denote the regular moment constant of rank $n$ corresponding to the function $g_i(t)$, i.e.,

$$\mu_{n,i} = \int_0^1 t^n dg_i(t).$$

Two regular Hausdorff summability methods given by $g_1(t)$ and $g_2(t)$ are equivalent if and only if $\mu_{n,1}/\mu_{n,2}$ and $\mu_{n,2}/\mu_{n,1}$ are both regular moment constants (e.g., see [3, p. 262]).

If, then, $g_1(t) = t^{\alpha_1}$ and $g_2(t) = t^{\alpha_2}$, we must show that

$$\frac{\mu_{n,1}}{\mu_{n,2}} = \frac{(1 + n/\alpha_2)}{(1 + n/\alpha_1)}$$

and that $\mu_{n,2}/\mu_{n,1}$ are both regular moment constants. This, however, is true [3, p. 264] and the proof is complete.

## 5. Positive definite functions.

LEMMA 13. *Let $f(x) \in L^1(-\infty, \infty)$. $f(x) \geq 0$ almost everywhere if and only if $f^*(x)$, the Fourier transform of $f(x)$, is positive definite.*

**Proof.**   Suppose that $f(x) \geqq 0$ almost everywhere.

$$(9) \qquad\qquad f*(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ixt} f(t) \, dt.$$

Since $f(x) \in L^1(-\infty, \infty)$ and is nonnegative almost everywhere,

$$\alpha(t) = \int_{-\infty}^{t} f(y) \, dy$$

is an increasing bounded function of $t$. Equation (9) may now be written as:

$$f*(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ixt} d\,\alpha(t),$$

which, by Bochner's theorem [2, p. 59], implies that $f*(x)$ is positive definite.
Suppose now that $f*(x)$ is positive definite. If

$$p(x) = \begin{cases} 1 - |x|/R, & |x| \leqq R, \\ 0, & |x| > R, \end{cases}$$

then

$$p*(x) = \frac{2}{\pi} \cdot \frac{(1 - \cos Rx)}{Rx^2}.$$

Thus, $p*(x) \in L^1$ and is positive. Since $p(x) \in L^1$ and is of bounded variation,

$$p(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ixt} p*(t) \, dt \qquad \text{(see [2, p. 12])}.$$

Hence, as above, $p(x)$ is positive definite. Since $f(x) \in L^1$, it is known [2, p. 14] that almost everywhere

$$f(x) = \lim_{R \to \infty} (2\pi)^{-1/2} \int_{-R}^{R} e^{-ixt} \left[ 1 - \frac{|t|}{R} \right] f*(t) \, dt$$

$$= \lim_{R \to \infty} (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ixt} p(t) f*(t) \, dt.$$

$p(t) \cdot f*(t)$, the product of two positive definite functions, is positive definite and belongs to $L^1$ for every $R$. Hence, almost everywhere, $f(x)$ is the limit of positive functions (see [2, p. 26]). This completes the proof.
From (4),

$$F'(A) = \int_{0}^{1} \cos At \, g(t) \, dt - (\sin A)/A = -\int_{0}^{1} \cos At[1 - g(t)] \, dt.$$

Since $F(0) = \pi/2$ and $F(\infty) = 0$, $F(A)$ is positive for all $A \geqq 0$ if $F'(A)$ is

negative for all $A > 0$. To find the conditions on $g(t)$ for this to occur, we recall our definition: $g(t) = 1$ for $t > 1$ and $g(-t) = g(t)$. Therefore,

$$F'(A) = -\frac{1}{2} \int_{-\infty}^{\infty} e^{iAt}[1 - g(t)]\, dt.$$

It follows from [2, p. 26] that $\int_{-\infty}^{\infty} e^{iAt}[1 - g(t)]\, dt \geqq 0$ if $1 - g(t)$ is positive definite. Lemma 1 now gives

THEOREM 10.   $g(t)$ *will not exhibit the Gibbs phenomenon if* $1 - g(t)$ *is positive definite.*

It has been shown elsewhere [6] that, of all regular Hausdorff means, this class and only this class will not exhibit the generalized Gibbs phenomenon.

Choose a function $f^*(A)$ with the properties:

(10)

$$
\begin{aligned}
&1. \quad f^*(A) \in L^2(-\infty, \infty), \\
&2. \quad f^*(A) \geqq 0 \quad \text{for } A \geqq 0, \\
&3. \quad f^*(-A) = -f^*(A) \quad \text{for } A > 0.
\end{aligned}
$$

Since $F(A)$ is odd and continuous, $g(t)$ will not exhibit the Gibbs phenomenon if and only if $H^*(A) \geqq 0$ almost everywhere, where

(11)
$$H^*(A) = 2f^*(A) \cdot F(A).$$

Recalling that $h(t) = g(t)/t$, we may write

$$F(A) = -(i/2) \int_{-\infty}^{\infty} e^{iAt} h(t)\, dt.$$

Let $h(t) \in L^2(0, 1)$. Since $h(t) \in L^2(1, \infty)$ and $h(-t) = -h(t)$, $F(A) \in L^2(-\infty, \infty)$. By the Schwarz inequality, $H^*(A) \in L^1$ since $f^*(A) \in L^2$. Thus, by Lemma 13, $H^*(A)$ is positive almost everywhere if and only if its Fourier transform, $H(x)$, is positive definite. By Parseval's theorem [11, p. 70], $H(x) = \int_{-\infty}^{\infty} f(x + t) h(t)\, dt$, where $if(t)$ is the Fourier transform of $f^*(t)$. We have, then,

THEOREM 11.   *Let* $h(t) = g(t)/t \in L^2(0, 1)$, *and let* $f(t) \in L^2(-\infty, \infty)$ *be any odd function such that the Fourier transform,* $f^*(t)$, *of* $-if(t)$ *satisfies conditions* (10). $g(t)$ *will exhibit the Gibbs phenomenon if and only if*

(12)
$$H(x) = \int_{-\infty}^{\infty} f(x + t)\, h(t)\, dt$$

*is not positive definite.*

In order that a function $H(x)$ be positive definite, it is necessary that $H(0) \geqq |H(x)|$ for all $x$ [2, p. 60]. This condition alone suffices to determine whether or not a given $g(t)$ exhibits the phenomenon.

THEOREM 12. *Suppose that $g(t)$, with $h(t) \in L^2(0,1)$, exhibits the Gibbs phenomenon. It is possible to find an $f(t)$ so that the function $H(x)$ defined in Theorem 11 does not attain its maximum at $x = 0$.*

**Proof.** Since $g(t)$ exhibits the phenomenon, $F(A)$ must be negative for all $A$ in some interval $0 \leq a \leq A \leq b$. Define $f^*(A)$ as follows:

$$f^*(A) = \begin{cases} 1, & a \leq A \leq b, \\ -1, & -b \leq A \leq -a, \\ 0, & \text{elsewhere.} \end{cases}$$

$f^*(A)$ satisfies conditions (10), $H^*(A)$ (11) is always negative and, therefore, $-H(x)$ is positive definite by Lemma 13. Thus, the maximum is not attained at $x = 0$.

LEMMA 14. *Let $f(t) \in L^1(-\infty, \infty)$ be positive decreasing for $t \geq 0$ and odd. Then $f^*(A) = -i \int_{-\infty}^{\infty} e^{iAt} f(t)\, dt$ is odd and $\geq 0$ for $A \geq 0$.*

**Proof.**

$$(13) \qquad\qquad f^*(A) = 2 \int_0^\infty \sin At\, f(t)\, dt.$$

Pólya [8, p. 378] proved (13) is nonnegative for $A \geq 0$ if $f(t)$ is positive decreasing for $t \geq 0$.

This lemma alone provides an abundance of $f(t)$ which may be used in the formulation of $H(x)$, for $f^*(A)$ will satisfy conditions (10), if, in addition, $f(t) \in L^2(-\infty, \infty)$. For example, the following two choices of $f(t)$ will be shown to yield much additional information.

$$(14) \qquad\qquad f(t) = \begin{cases} a - t, & 0 \leq t \leq a, \\ -(a + t), & -a \leq t < 0, \\ 0, & |t| > a. \end{cases}$$

$$(15) \qquad\qquad f(t) = \begin{cases} 1, & 0 \leq t \leq a, \\ -1, & -a \leq t < 0, \\ 0, & |t| > a. \end{cases}$$

THEOREM 13. *Let $h(t) \in L^2(0,1)$. $g(t)$ will exhibit the Gibbs phenomenon if, for any value of $a > 0$,*

$$-2aK(x) + \int_0^{x+a} K(t)\, dt + \int_0^{a-x} K(t)\, dt$$

*is convex in some neighborhood of $x = 0$, where $K(t) = \int_0^t h(y)\, dy$.*

**Proof.** Using (14) as our $f(t)$, we find that (12) becomes

$$H(x) = \int_0^{a-x} (a - t - x)h(t)\, dt - \int_0^x (a + t - x)h(t)\, dt + \int_x^{x+a} (a - t + x)h(t)\, dt.$$

Integration by parts yields

$$(16) \qquad H(x) = -2aK(x) + \int_0^{x+a} K(t)\,dt + \int_0^{a-x} K(t)\,dt.$$

If $H(x)$ is convex in a neighborhood of $x = 0$, it is not positive definite, and the proof follows from Theorem 11.

THEOREM 14.  *Let* $h(0) = 0$, *let* $h(x)$ *be continuous in some neighborhood of* $x = a > 0$, *and let* $h'(x)$ *exist in some right-hand neighborhood of* $x = 0$. $g(t)$ *will exhibit the Gibbs phenomenon if*

$$h'(x) < (1/2a)[h(x + a) + h(a - x)]$$

*in some right-hand neighborhood of* $x = 0$.

**Proof.**  For values of $x$ in some right-hand neighborhood of $x = 0$, we get from (16) that

$$H'(x) \; = \; -2ah(x) + K(x + a) - K(a - x),$$
$$H''(x) \; = \; -2ah'(x) + h(x + a) + h(a - x).$$

By hypothesis, $H''(x)$ is positive for all $x$ in some right-hand neighborhood of $x = 0$ and $H'(0) = 0$. Consequently, $H(x)$ is convex in some neighborhood of $x = 0$, and the proof is completed by the preceding theorem.

Theorem 8 may be used to show that Theorem 14 is not true if either of the conditions, $h(0) = 0$ and $h'(x) < (1/2a)[h(x + a) + h(a - x)]$, is removed.

THEOREM 15.  *Let* $h(t) \in L^2(0, 1)$. $g(t)$ *will exhibit the Gibbs phenomenon if, for any value of* $a > 0$,

$$K(a - x) + K(a + x) - 2K(x)$$

*is convex in some neighborhood of* $x = 0$, *where* $K(x) = \int_0^x h(t)\,dt$.

**Proof.**  Use (15) in (12).  Then

$$(17) \qquad\qquad H(x) = K(a - x) + K(a + x) - 2K(x),$$

and the proof follows as in Theorem 13.

The next theorem follows from this as Theorem 14 followed from Theorem 13.

THEOREM 16.  *Let* $h(0) = 0$, *and let* $h'(x)$ *exist in some right-hand neighborhood of* $x = 0$ *and some neighborhod of* $x = a > 0$. $g(t)$ *will exhibit the Gibbs phenomenon if*

$$h'(x) < \frac{1}{2}[h'(a - x) + h'(a + x)]$$

*in some right-hand neighborhood of* $x = 0$.

We now present two theorems which, unlike every other theorem encountered so far, do not require any statement concerning continuity, discontinuity, bounded variation, differentiability, or order of zero.

THEOREM 17. *Let* $h(t) \in L^2(0,1)$. $g(t)$ *will exhibit the Gibbs phenomenon if, for some* $a > 0$, $\int_0^a h(t)\, dt \leq 0$.

**Proof.** Using (17), we get $H(0) \leq 0$. Therefore $H(x)$ is not positive definite.

THEOREM 18. *Let* $h(t) \in L^2(0,1)$. $g(t)$ *will exhibit the Gibbs phenomenon if, for some* $a > 0$,

$$(18) \qquad\qquad \int_0^{2a} h(t)\, dt > 4 \int_0^a h(t)\, dt.$$

**Proof.** It follows from (17) and (18) that $H(a) > H(0)$. Hence, $H(x)$ is not positive definite.

**6. Remark.** We recall that Theorem 3 required the apparently peculiar condition that $h'(1) \neq -1$. We are now in a position to show that this condition is by no means peculiar, and that we may improve upon the theorem.

Let us again use (14) in (12). Then, $H''(x) = h(a - x) + h(a + x) - 2ah'(x)$. Suppose that $h(0) = 0$ and that $h'(x)$ is continuous in $0 \leq x \leq 1$. (Note that, unlike Theorem 3, $h'(x)$ is not required to be of bounded variation.) If $h'(1 - ) = -1$ (in Theorem 3, $h(x)$ was not defined for $|x| > 1$ so that $h'(1)$ meant the one-sided derivative), then since $h(x) = 1/x$ for $x > 1$ (and $h(-x) = -h(x)$), $h'(x)$ will be continuous and bounded in $-\infty < x < \infty$. Then, $H''(x)$ and $H(x)$ belong to $L^1$ and are continuous and bounded in $(-\infty, \infty)$. Bochner [1, p. 96] has shown that, under these circumstances, $H(x)$ is positive definite if and only if $-H''(x)$ is positive definite. Hence, if $h'(1 - ) = -1$, there is the possibility that $-H''(x)$ is positive definite and, therefore, so is $H(x)$. Consequently, $g(t)$ will not exhibit the Gibbs phenomenon.

We may, however, include the following theorem as a companion to Theorem 3.

THEOREM 19. *Let* $h'(x)$ *exist and be continuous and bounded in*

$$-\infty < x < \infty,$$

*and let* $h(0) = 0$. $g(t)$ *will exhibit the Gibbs phenomenon if* $h'(0) \leq h(a)/a$.

Note that here we permit $h'(0) \leq h(a)/a$ while in Theorem 14, the strict inequality was required. We may do so since $h'(0) = h(a)/a$ implies $-H''(0) = 0$ so that $-H''(x)$ cannot be positive definite.

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YESHIVA UNIVERSITY,
    NEW YORK, NEW YORK